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Exact results on twist anomaly

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Abstract: In vacuum string field theory, the sliver state solution has been proposed as a candidate of a D-brane configuration. Physical observables associated with this solution, such as its energy density and the tachyon mass, are written in terms of the Neumann coefficients. These observables, though vanish naively due to twist symmetry, acquire non-vanishing values arising from their singular behavior. Therefore, this phenomenon is called twist anomaly. In this paper we present an analytical derivation of these physical observables with the help of the star algebra spectroscopy. We also identify in our derivation the origin of the twist anomaly in the finite-size matrix regularization.

Keywords: Bosonic Strings, D-branes, Anomalies in Field and String Theories.

Contents

1. Introduction

Vacuum string field theory (VSFT) [1]–[4] is a candidate for the string field theory expanded around the tachyon vacuum. Since there are no open string excitations in the tachyon vacuum, the VSFT conjecture claims that the string field theory around the tachyon vacuum is obtained by replacing the BRST charge in cubic string field theory (CSFT) [5] by a purely ghost operator. Although this sounds persuasive, we would like to know whether this conjecture is true. Hence, it is very important to construct the D25-brane solution explicitly in VSFT and investigate the energy density of the solution and the fluctuation spectrum around it.

A translationally and Lorentz invariant classical solution of VSFT has been found in [6] following earlier investigations [2, 7]. Because this solution is believed to correspond to a D25-brane, it is expected that the energy density of the solution is equal to the D25-brane tension and the fluctuation spectrum around the solution reproduces that of the ordinary open string theory. To investigate the energy density and the fluctuation spectrum of the solution, in [6] a tachyon wave function was constructed and the mass spectrum and the potential height are given in terms of the Neumann coefficients expressing the three-string interactions. By the numerical study of [6, 8], it was found that, although the tachyon mass is correctly reproduced, contrary to the expectation, the energy density is about twice the D25-brane tension.

Furthermore, in [8] they evaluated these physical observables by using various identities among the Neumann coefficients. The result is as follows. A Neumann coefficient matrix has a twist symmetry, which causes the degeneracy of its eigenvalues. Due to this degeneracy, the main constituents of the physical observables seem to vanish naively. However, actually in the numerical analysis, the observables acquire non-vanishing values, since the expressions of the physical observables behave singularly at the end of the eigenvalue distribution of the Neumann coefficient [8, 9, 10]. This is interpreted as a breakdown of the twist symmetry and hence this phenomenon is called "twist anomaly". Since the calculations in [6, 8] have been done only by numerical method, it should be very interesting if we could evaluate the twist anomaly exactly. For such analytic calculation, we have to solve the eigenvalue problem of the Neumann coefficient matrices.

On the other hand, these observables in VSFT are computed exactly by using boundary conformal field theory (BCFT) technique [10]. Using BCFT, they showed that the tachyon mass squared is exactly equal to $-1/\alpha'$ and the ratio of the energy density to the D25-brane tension is given by $(\pi^2/3)[16/(27\ln 2)]^3 \simeq 2.05\cdots$, instead of the integer 2. Since the correspondence between these geometric method of BCFT and the algebraic one using the Neumann coefficients is still not explicit, it is worth repeating exact calculations by the algebraic method.

Recently, an elegant paper [11] has appeared, which solves the eigenvalue problem of the Neumann coefficient matrix (star algebra spectroscopy). In this paper, using the results of [11], we compute analytically the observables such as the tachyon mass and the energy density by the algebraic method to obtain the same results as those by the BCFT method. In our analysis we adopt the regularization by cutting off the size of the Neumann coefficient matrices to finite size ones. In particular, we clarify the origin of the twist anomaly in this regularization. Other applications of the star algebra spectroscopy are found in [12, 13, 14].

The organization of the rest of this paper is follows. In the next section, we review the classical solution of the VSFT action, physical observables of this solution and their interpretation as twist anomaly. In section 3, we introduce the finite-size matrix regularization of the Neumann coefficient matrices and calculate the tachyon mass by using this regularization. Section 4 is devoted to the evaluation of the energy density of the solution. In the final section, we summarize the paper and discuss some future problems. In appendix A and B, we present some technical details used in the text.

2. VSFT and its observables

In this section, we shall review how the physical observables are written in terms of the Neumann coefficients. For this purpose, we shall first summarize the VSFT action and the properties of the Neumann coefficients, and then proceed to the construction of a classical solution and observables associated with the solution.

2.1 VSFT action and the Neumann coefficients

The action of VSFT is given by [1, 2, 4]

$$\mathcal{S}[\Phi] = -K\left(\frac{1}{2}\Phi\cdot\mathcal{Q}\Phi + \frac{1}{3}\Phi\cdot(\Phi*\Phi)\right), \tag{2.1}$$

where K is a constant and $\mathcal{Q}$ is the purely ghost BRST operator around the tachyon vacuum. The star product $*$ is the same as in ordinary CSFT and defined through the three-string vertex,

$$\begin{aligned}|V\rangle_{123} &= \exp\left(-\frac{1}{2}\mathbf{A}^\dagger C\mathcal{M}_3\mathbf{A} - \mathbf{A}^\dagger\mathbf{V} - \frac{1}{2}V_{00}(A_0)^2 + (\text{ghost part})\right)|0\rangle_{123}\times\\ &\quad\times(2\pi)^{26}\delta^{26}(p_1+p_2+p_3)\,,\end{aligned} \tag{2.2}$$

with various quantities defined by $(n, m \geq 1)$

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} a_n^{(1)}\\ a_n^{(2)}\\ a_n^{(3)}\end{pmatrix}, \qquad A_0 = \begin{pmatrix} a_0^{(1)}\\ a_0^{(2)}\\ a_0^{(3)}\end{pmatrix}, \qquad \mathcal{M}_3 = \begin{pmatrix} M_0 & M_+ & M_-\\ M_- & M_0 & M_+\\ M_+ & M_- & M_0\end{pmatrix},\\ \mathbf{V} &= \begin{pmatrix} \mathbf{v}_0 & \mathbf{v}_+ & \mathbf{v}_-\\ \mathbf{v}_- & \mathbf{v}_0 & \mathbf{v}_+\\ \mathbf{v}_+ & \mathbf{v}_- & \mathbf{v}_0\end{pmatrix}A_0\,, \qquad V_{00} = \frac{1}{2}\ln\left(\frac{3^3}{2^4}\right), \qquad C_{nm} = (-1)^n\delta_{nm}\,.\end{aligned} \tag{2.3}$$

The matter oscillator $a_n^{(r)\mu}$ $(n \geq 1)$ is normalized to satisfy the commutation relation,

$$[a_n^{(r)\mu}, a_m^{(s)\nu\dagger}] = \eta^{\mu\nu}\delta_{nm}\delta^{rs}\,, \tag{2.4}$$

and $a_0^{(r)}$ is related to the center-of-mass momentum of the string r, $p_r = -i\partial/\partial x_r$, by $a_0^{(r)} = \sqrt{2}\,p_r$ (we are adopting the convention of $\alpha' = 1$). We follow the notations of [6, 8].

The Neumann coefficients M_0, $M_\pm$, $\mathbf{v}_0$ and $\mathbf{v}_\pm$ satisfy the following linear relations:

$$M_0 + M_+ + M_- = 1\,, \tag{2.5}$$

$$\mathbf{v}_0 + \mathbf{v}_+ + \mathbf{v}_- = 0\,, \tag{2.6}$$

$$CM_0C = M_0\,, \qquad C\mathbf{v}_0 = \mathbf{v}_0\,, \tag{2.7}$$

$$CM_\pm C = M_\mp\,, \qquad C\mathbf{v}_\pm = \mathbf{v}_\mp\,. \tag{2.8}$$

It is convenient to introduce a new twist-odd matrix M_1 and vector $\mathbf{v}_1$,

$$M_1 = M_+ - M_-\,, \tag{2.9}$$

$$\mathbf{v}_1 = \mathbf{v}_+ - \mathbf{v}_-\,, \tag{2.10}$$

$$CM_1C = -M_1\,, \qquad C\mathbf{v}_1 = -\mathbf{v}_1\,, \tag{2.11}$$

and regard twist-even Neumann coefficients M_0 and $\mathbf{v}_0$ and twist-odd ones M_1 and $\mathbf{v}_1$ as independent. Then, these Neumann coefficients are also known to satisfy the following

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non-linear identities [15, 16, 17]:

$$[M_0, M_1] = 0\,, \tag{2.12}$$

$$M_1^2 = (1 - M_0)(1 + 3M_0)\,, \tag{2.13}$$

$$3(1 - M_0)\mathbf{v}_0 + M_1\mathbf{v}_1 = 0\,, \tag{2.14}$$

$$3M_1\mathbf{v}_0 + (1 + 3M_0)\mathbf{v}_1 = 0\,, \tag{2.15}$$

$$\frac{9}{4}\mathbf{v}_0^2 + \frac{3}{4}\mathbf{v}_1^2 = 2\,V_{00}\,. \tag{2.16}$$

2.2 Classical solution, tachyon mode and observables

The equation of motion of VSFT is given by

$$\mathcal{Q}\Psi_{\rm c} + \Psi_{\rm c} * \Psi_{\rm c} = 0\,. \tag{2.17}$$

Adopting the Siegel gauge for Ψ_c, $|\Psi_c\rangle = b_0|\phi_{\rm c}\rangle$, and the squeezed state ansatz for $|\phi_{\rm c}\rangle$ [2, 6, 7]:

$$|\phi_{\rm c}\rangle = \mathcal{N}_{\rm c}\exp\left(-\frac{1}{2}\sum_{n,m\geq 1} a_n^\dagger (CT)_{nm} a_m^\dagger + \sum_{n,m\geq 1} c_n^\dagger (C\widetilde{T})_{nm} b_m^\dagger\right)|0\rangle\,, \tag{2.18}$$

the equation of motion (2.17) is satisfied by choosing the real matrix T_{nm} as

$$T = \frac{1}{2M_0}\left(1 + M_0 - \sqrt{(1 - M_0)(1 + 3M_0)}\right), \tag{2.19}$$

and the normalization factor $\mathcal{N}_{\rm c}$ as

$$\mathcal{N}_{\rm c} = -\left[\det(1 - T\mathcal{M})\right]^{13}\left[\det(1 - \widetilde{T}\widetilde{\mathcal{M}})\right]^{-1}, \tag{2.20}$$

with

$$\mathcal{M} = \begin{pmatrix} M_0 & M_+ \\ M_- & M_0 \end{pmatrix}. \tag{2.21}$$

The Neumann coefficients with a tilde are those for the ghost part. The form of the BRST operator $\mathcal{Q}$ is uniquely fixed by the requirement of the existence of the solution under the above assumption [6, 12, 18, 19]. The identification of the state (2.18) with T given by (2.19) as the sliver state [2, 20] has recently been proved by [14].

To see whether this solution corresponds to a D25-brane, we have to examine whether the fluctuation spectrum and the energy density of this solution give the expected ones. As the first step of the examination of the mass spectrum, we solve the linearized equation of motion for the tachyon wave function Φ_t,

$$\mathcal{Q}_{\rm B}\Phi_t \equiv \mathcal{Q}\Phi_{\rm t} + \Psi_{\rm c} * \Phi_{\rm t} + \Phi_{\rm t} * \Psi_{\rm c} = 0\,, \tag{2.22}$$

where $\mathcal{Q}_{\rm B}$ is the BRST operator for fluctuations around $\Psi_{\rm c}$. A plausible choice for the tachyon fluctuation mode $|\Phi_t\rangle = b_0|\phi_{\rm t}\rangle$ in the Siegel gauge was proposed in [6]:

$$|\phi_{\rm t}\rangle = \frac{\mathcal{N}_{\rm t}}{\mathcal{N}_{\rm c}}\exp\left(-\sum_{n\geq 1} t_n a_n^\dagger a_0\right)|\phi_{\rm c}\rangle\,. \tag{2.23}$$

$1/\sqrt{1+3M_0}$	M_1	$\mathbf{v}_0$	$\mathbf{v}_1$	$\mathbf{t}$
1	-1	0	1	1

Table 1: Degrees of singularity for various quantities.

It was found that the linearized equation of motion (2.22) is satisfied if the vector $\mathbf{t}$ is given by

$$\mathbf{t} = 3(1+T)(1+3M_0)^{-1}\mathbf{v}_0 \,, \tag{2.24}$$

and the center-of-mass momentum $p^\mu = a_0^\mu/\sqrt{2}$ satisfies the following on-shell condition:

$$p^2 = -m_t^2 \equiv \frac{\ln 2}{G} \,. \tag{2.25}$$

Here G is given in terms of the Neumann coefficients as follows:

$$\begin{aligned} G =\, & 2V_{00} + (\mathbf{v}_- - \mathbf{v}_+, \mathbf{v}_+ - \mathbf{v}_0)(1-T\mathcal{M})^{-1}T\begin{pmatrix}\mathbf{v}_+ - \mathbf{v}_- \\ \mathbf{v}_- - \mathbf{v}_0\end{pmatrix} + \\ & + 2(\mathbf{v}_- - \mathbf{v}_+, \mathbf{v}_+ - \mathbf{v}_0)(1-T\mathcal{M})^{-1}\begin{pmatrix}0\\ \mathbf{t}\end{pmatrix} + (0,\mathbf{t})\mathcal{M}(1-T\mathcal{M})^{-1}\begin{pmatrix}0\\ \mathbf{t}\end{pmatrix} . \end{aligned} \tag{2.26}$$

The normalization factor $\mathcal{N}_\mathrm{t}$ for Φ_t in (2.23) is determined to be

$$\mathcal{N}_\mathrm{t} = \frac{1}{\sqrt{KG}}\left[\det(1-T^2)\right]^{13/2}\left[\det(1-\widetilde{T}^2)\right]^{-1/2}\exp\left(\mathbf{t}\,(1+T)^{-1}\mathbf{t}\, m_\mathrm{t}^2\right) , \tag{2.27}$$

from the requirement that Φ_t has a canonical kinetic term:

$$\frac{K}{2}\Phi_\mathrm{t}\cdot\mathcal{Q}_\mathrm{B}\Phi_\mathrm{t} \underset{p^2\sim -m_\mathrm{t}^2}{\sim} -\frac{1}{2}(p^2+m_\mathrm{t}^2) \,. \tag{2.28}$$

As the first test of the present classical solution Ψ_c and tachyon mode Φ_t, we have to check whether the tachyon mass m_t^2 (2.25) reproduces the correct value of -1. It was found numerically in [6] that, by truncating the infinite-size Neumann coefficient matrices into finite but large ones, the quantity G actually gives the expected value $\ln 2$. Later in [8], an interesting interpretation of G (and other observables in VSFT) was presented. They found that this quantity G (2.26) vanishes identically, if we naively use the various non-linear relations among the Neumann coefficients, (2.12)–(2.14). The vanishing of G can be ascribed to that the eigenvalues of the matrix M_0 are doubly degenerate between twist-even and odd eigenvectors. This phenomenon that the quantity G which vanishes naively due to twist symmetry can actually have non-vanishing value is called "twist anomaly" in [8]. Further in [8], from the numerical analysis they found that this paradox emerges because the expression of G (2.26) is singular at $M_0 = -1/3$ (actually the eigenvalue distribution of M_0 ranges between $-1/3$ and 0 [8, 9, 11]). Due to this singularity, the expression (2.26) takes a form of the difference of two divergent quantities, and hence a careful treatment is necessary. On the basis of numerical analysis, the following calculation rules leading to correct values has been proposed:

- We assign each Neumann coefficient matrix its degree of singularity as given in table 1. This assignment is compatible with the non-linear relations (2.13)–(2.15).

- We laurent-expand the quantity such as G around the singular point $M_0 = -1/3$.
- For the terms with degrees of singularity less than three, we can freely use the non-linear relations (2.12)–(2.16).
- For the most singular terms with degree three, we treat them as they stand. However, we are allowed to use the non-linear relation (2.14) to express $\mathbf{v}_0$ in terms of $\mathbf{v}_1$; $\mathbf{v}_0 = -(1/3)(1-M_0)^{-1}M_1\mathbf{v}_1$.

Using this rule, we find that the quantity G (2.26) is simplified into the following form:

$$G = -\frac{9\sqrt{3}}{32}\mathbf{v}_1\left(M_1\frac{1}{(1+3M_0)^{3/2}}M_1 - \frac{1-M_0}{\sqrt{1+3M_0}}\right)\mathbf{v}_1\,. \tag{2.29}$$

In the next section we shall regularize this indefinite expression properly and evaluate it analytically.

Now let us turn to the energy density of the classical solution. First, the energy density $\mathcal{E}_{\mathrm{c}}$ of the solution Ψ_{c} is given by

$$\mathcal{E}_{\mathrm{c}} = -\frac{\mathcal{S}[\Psi_{\mathrm{c}}]}{V_{26}} = \frac{K}{6}\left(\frac{[\det(1-T\mathcal{M})]^2}{\det(1-T^2)}\right)^{13}\left(\frac{[\det(1-\widetilde{T}\widetilde{\mathcal{M}})]^2}{\det(1-\widetilde{T}^2)}\right)^{-1}, \tag{2.30}$$

For comparing $\mathcal{E}_{\mathrm{c}}$ with the D25-brane tension T_{25}, let us calculate the latter. It is given in the present convention of $\alpha' = 1$ by $T_{25} = 1/(2\pi^2 g_{\mathrm{o}}^2)$ with g_{o} being the open string coupling constant defined as the three-tachyon on-shell amplitude. Using the tachyon wave function Φ_{t} (2.23), g_{o} is given by

$$\begin{aligned} g_{\mathrm{o}} &= K\,\Phi_{\mathrm{t}}\cdot(\Phi_{\mathrm{t}} * \Phi_{\mathrm{t}})\big|_{p_1^2=p_2^2=p_3^2=-m_{\mathrm{t}}^2} \\ &= K\mathcal{N}_{\mathrm{t}}^3\,[\det(1-T\mathcal{M}_3)]^{-13}\det(1-\widetilde{T}\widetilde{\mathcal{M}}_3)\times \\ &\quad\times\exp\left\{-\frac{1}{2}\mathbf{V}(1-T\mathcal{M}_3)^{-1}TC\mathbf{V} + \mathbf{V}(1-T\mathcal{M}_3)^{-1}\mathbf{t}A_0 - \right. \\ &\qquad\left. -\frac{1}{2}A_0\mathbf{t}\mathcal{M}_3(1-T\mathcal{M}_3)^{-1}\mathbf{t}A_0 - \frac{1}{2}V_{00}(A_0)^2\right\}. \end{aligned} \tag{2.31}$$

From (2.27), (2.30) and (2.31), we find the expression for the ratio $\mathcal{E}_{\mathrm{c}}/T_{25}$:

$$\frac{\mathcal{E}_{\mathrm{c}}}{T_{25}} = \frac{\pi^2}{3G^3}\exp(6m_{\mathrm{t}}^2 H)\,, \tag{2.32}$$

where H is given by

$$\begin{aligned} H = -\frac{2}{(A_0)^2}\Big[&-\frac{1}{2}\mathbf{V}(1-T\mathcal{M}_3)^{-1}TC\mathbf{V} + \mathbf{V}(1-T\mathcal{M}_3)^{-1}\mathbf{t}A_0 - \\ &-\frac{1}{2}A_0\mathbf{t}\mathcal{M}_3(1-T\mathcal{M}_3)^{-1}\mathbf{t}A_0\Big] + \mathbf{t}(1+T)^{-1}\mathbf{t} + V_{00}\,. \end{aligned} \tag{2.33}$$

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Following the above calculation rules, we can simplify (2.33) into

$$\begin{aligned}H =\ & \frac{\sqrt{3}}{4}\mathbf{v}_1\frac{1}{\sqrt{1+3M_0}}\mathcal{R}\mathbf{v}_1 - \frac{3\sqrt{3}}{8}\mathbf{v}_1\frac{1}{\sqrt{1+3M_0}}M_1\frac{1}{\sqrt{1+3M_0}}\mathcal{R}\frac{1}{\sqrt{1+3M_0}}M_1\mathbf{v}_1 - \\ & -\frac{9\sqrt{3}}{16}\mathbf{v}_1M_1\frac{1}{1+3M_0}\mathcal{R}\frac{1}{\sqrt{1+3M_0}}M_1\mathbf{v}_1 + \frac{9\sqrt{3}}{16}\mathbf{v}_1M_1\frac{1}{(1+3M_0)^{3/2}}M_1\mathbf{v}_1 - \\ & -\frac{\sqrt{3}}{16}\mathbf{v}_1\frac{(1+3M_0)^{3/2}}{5-M_0}\mathbf{v}_1\,, \end{aligned} \tag{2.34}$$

where $\mathcal{R}$ is defined by

$$\mathcal{R} = \left(1+\frac{1}{4}M_1\frac{1}{\sqrt{1+3M_0}}M_1\frac{1}{\sqrt{1+3M_0}}\right)^{-1}. \tag{2.35}$$

Analytic evaluation of H will be given in section 4.

3. Tachyon mass

Let us proceed to evaluating the tachyon mass or the quantity G analytically. From the numerical analysis in [6, 8] we know that the quantity G gives the expected value $\ln 2$ to high precision. We shall show analytically that this is exact. The analysis in this section will also reveal how the breakdown of twist symmetry brought about by regularization makes G non-vanishing.

3.1 Finite-size matrix regularization

Before proceeding to evaluating the tachyon mass exactly, we shall explain how to regularize the expression (2.29) of G which is indefinite due to the eigenvalue $-1/3$ of M_0. For this purpose we shall first present expressions of the Neumann coefficient matrices in terms of simpler quantities.

In [11] it was found that the Neumann coefficient matrices M_0 and M_1 are related to another simpler matrix K_1, which is the matrix representation of the Virasoro algebra, $K_1 = L_1 + L_{-1}$, by

$$M_0 = -\frac{1}{1+2\cosh(K_1\pi/2)}\,, \tag{3.1}$$

$$M_1 = \frac{2\sinh(K_1\pi/2)}{1+2\cosh(K_1\pi/2)}\,. \tag{3.2}$$

The matrix representation of K_1 can be read off from the operation of the Virasoro algebra:

$$(K_1)_{nm} = -\sqrt{(n-1)n}\,\delta_{n-1,m} - \sqrt{n(n+1)}\,\delta_{n+1,m}\,. \tag{3.3}$$

Note that K_1 is symmetric and twist-odd:

$$K_1^T = K_1\,, \qquad CK_1C = -K_1\,. \tag{3.4}$$

For the vectors $\mathbf{v}_0$ and $\mathbf{v}_1$, we have the following convenient expressions:

$$\mathbf{v}_0 = -\frac{1}{3}(1+3M_0)\mathbf{u}\,, \tag{3.5}$$

$$\mathbf{v}_1 = M_1\mathbf{u}\,, \tag{3.6}$$

with a new vector $\mathbf{u}$ defined as

$$u_n = \frac{1}{\sqrt{n}}\cos\left(\frac{n\pi}{2}\right) = \begin{cases} \frac{(-1)^{n/2}}{\sqrt{n}}\,, & n\text{: even}\,, \\ 0\,, & n\text{: odd}\,. \end{cases} \tag{3.7}$$

Proof of (3.5) and (3.6) is given in appendix A.

Now we can express G (2.29) in terms of the simple matrix K_1 and vector $\mathbf{u}$. Adopting the calculation rules in [8], which we summarized in section 2.2, all we have to do is to concentrate on the most singular term. Namely, we assign K_1 and $\mathbf{u}$ their degrees of singularity -1 and 2, respectively, and laurent-expand M_0, M_1 and $\mathbf{v}_1$ in G with respect to K_1 around the singularity $K_1 = 0$ corresponding to $M_0 = -1/3$:

$$1+3M_0 \simeq \frac{\pi^2}{12}K_1^2\,, \tag{3.8}$$

$$M_1 \simeq \frac{\pi}{3}K_1\,, \tag{3.9}$$

$$\mathbf{v}_1 \simeq \frac{\pi}{3}K_1\mathbf{u}\,. \tag{3.10}$$

Keeping only the terms with degree three, the quantity G can be expressed as[1]

$$G = \frac{\pi}{4}\left(\mathbf{u}^T K_1 \frac{1}{\sqrt{K_1^2}} K_1 \mathbf{u} - \mathbf{u}^T K_1 K_1 \left(\frac{1}{\sqrt{K_1^2}}\right)^3 K_1 K_1 \mathbf{u}\right). \tag{3.11}$$

The present expression of G is indefinite as ever, and we have to regularize it. In the numerical analysis of the original expression of G (2.29), we adopted the level truncation, namely we cut-off all the infinite dimensional matrices into $L\times L$ ones. While M_0 without regularization has the eigenvalue $-1/3$ [8, 9, 11], in the level truncation the lowest eigenvalue of M_0 is lifted from $-1/3$. Hence, this level truncation serves as a good regularization.

In this finite-size regularization, we have truncated the infinite-size matrices M_0 and M_1 into those of the same size L. Therefore, the regularized version of the matrix K_1^2 in the denominator of (3.11) should be the truncation of the square of the infinite dimensional matrix, $(K_1)^2|_L$, since this K_1^2 originates from $1+3M_0$ (3.8). On the other hand, the regularized version of K_1 in the numerator of (3.11), which comes from M_1 (3.9), should simply be the truncation of K_1 itself, $K_1|_L$.

3.2 Matrix representation

In this subsection, let us study how the degeneracy of the eigenvalues between twist-even and odd eigenvectors is lifted due to finite-size matrix regularization. Here we shall change rows and columns of a generic matrix M. We shall bring rows with odd indices into the upper side and ones with even indices into the lower side, and repeat a similar manipulation

[1] Since the r.h.s. of (3.11) manifestly vanishes by naive treatment, we do not need to add it the correction term $G_{\rm reg}$ with degree less than three (see [8, section 3.2]).

for columns:

$$M = \begin{pmatrix} M_{oo} & M_{oe} \\ M_{eo} & M_{ee} \end{pmatrix}. \tag{3.12}$$

Since the matrix K_1 is twist-odd, $CK_1C = -K_1$, its diagonal blocks vanish in this representation.

$$K_1|_L = \begin{pmatrix} 0 & (K_1)_{oe} \\ (K_1)_{eo} & 0 \end{pmatrix}. \tag{3.13}$$

Although the off-diagonal blocks are not symmetric matrices by themselves, we can give them a useful decomposition. First let us consider the case that the truncation level L is an even number 2ℓ. After squaring (3.13), we find the diagonal block symmetric. Therefore, we can diagonalize it:

$$(K_1|_{2\ell})^2 = \begin{pmatrix} P_{2\ell}\Lambda_{2\ell}^2 P_{2\ell}^T & 0 \\ 0 & Q_{2\ell}\Lambda_{2\ell}^2 Q_{2\ell}^T \end{pmatrix}, \tag{3.14}$$

where the diagonal matrix $\Lambda_{2\ell}$ and the orthogonal matrices $P_{2\ell}$ and $Q_{2\ell}$ are all $\ell \times \ell$ ones. Note here that the eigenvalues κ of the odd-odd sector and even-even one degenerate because their eigenvalue equations are identical:

$$\det\big((K_1)_{oe}(K_1)_{eo} - \kappa^2 1\big) = 0 \Longleftrightarrow \det\big((K_1)_{eo}(K_1)_{oe} - \kappa^2 1\big) = 0\,. \tag{3.15}$$

The eigenvalue distribution of finite-size regularized matrix K_1 was analyzed in [11]. They found that the spacing between the nearest eigenvalues is independent of κ and given by

$$\Delta|\kappa| = \frac{2\pi}{\ln L}\,. \tag{3.16}$$

Assuming that the spacing (3.16) applies also to two adjacent eigenvalues with opposite sign[2] and using the facts that the eigenvalues κ and $-\kappa$ are always paired and that there exits no zero eigenvalue in the case $L = 2\ell$, the eigenvalues of $K_1|_{2\ell}$, which are the diagonal elements of $\Lambda_{2\ell}$, are given by

$$(\Lambda_{2\ell})_{nn} = \frac{2\pi}{\ln L}\left(n - \frac{1}{2}\right) \equiv \kappa_{n-\frac{1}{2}}\,. \tag{3.17}$$

Note that the degeneracy in (3.14) always occurs for any finite-size matrix M which is symmetric $M^T = M$ and twist-odd $CMC = -M$.

Using the expression (3.14) of $(K_1|_{2\ell})^2$, we find a useful decomposition for K_1.

$$K_1|_{2\ell} = \begin{pmatrix} 0 & P_{2\ell}\Lambda_{2\ell}Q_{2\ell}^T \\ Q_{2\ell}\Lambda_{2\ell}P_{2\ell}^T & 0 \end{pmatrix}. \tag{3.18}$$

This decomposition is unique up to the overall sign. This can be seen by counting the degrees of freedom in the matrix. Each of the $\ell \times \ell$ orthogonal matrices $P_{2\ell}$ and $Q_{2\ell}$ has $\ell(\ell-1)/2$ degrees of freedom, and the diagonal matrix $\Lambda_{2\ell}$ has ℓ degrees of freedom. Therefore, the number of degrees of freedom in the r.h.s. of (3.18) is equal to $\ell(\ell-1)/2 \times 2 + \ell = \ell^2$, which agrees with that of $(K_1)_{oe} = [(K_1)_{eo}]^T$ on the r.h.s. of (3.13). Since the expression (3.18) has the same number of degrees of freedom as in the original matrix (3.13), we see that the decomposition is unique.

[2]Numerical analysis of the smallest solution κ in [11, (6.3)] supports this assumption to high precision.

The same argument holds for the case that the truncation level L is an odd number $2\ell+1$, except that in this case we encounter rectangular matrices and a careful analysis is necessary. In this case the square of the matrix K_1 is

$$(K_1|_{2\ell+1})^2 = \begin{pmatrix} P_{2\ell+1}\,\mathrm{diag}(\Lambda^2_{2\ell+1},0)P^T_{2\ell+1} & 0 \\ 0 & Q_{2\ell+1}\Lambda^2_{2\ell+1}Q^T_{2\ell+1} \end{pmatrix}. \tag{3.19}$$

Here $\Lambda_{2\ell+1}$ and $Q_{2\ell+1}$ are $\ell\times\ell$ matrices, while $P_{2\ell+1}$ is a $(\ell+1)\times(\ell+1)$ matrix. Since both the odd-odd block (an $(\ell+1)\times(\ell+1)$ matrix) and the even-even one (an $\ell\times\ell$ matrix) have the same rank, the former one should have an extra zero eigenvalue. The diagonal matrix $\Lambda_{2\ell+1}$ is now given as

$$(\Lambda_{2\ell+1})_{nn} = \frac{2\pi}{\ln L}n \equiv \kappa_n\,. \tag{3.20}$$

The eigenvalue spectrum (3.20) can be understood from the spacing (3.16) and the fact that we have a zero eigenvalue in the present case. Hence, the matrix $K_1|_{2\ell+1}$ by itself reads

$$K_1|_{2\ell+1} = \begin{pmatrix} 0 & P_{2\ell+1}(\Lambda_{2\ell+1},\mathbf{0})^T Q^T_{2\ell+1} \\ Q_{2\ell+1}(\Lambda_{2\ell+1},\mathbf{0})P^T_{2\ell+1} & 0 \end{pmatrix}, \tag{3.21}$$

where $(\Lambda_{2\ell+1},\mathbf{0})$ is an $\ell\times(\ell+1)$ matrix with vanishing $(\ell+1)$-th column.

Now let us return to the expression (3.11) of G. As we explained in the previous subsection, the regularized version of K_1^2 in the denominator stands for $(K_1)^2|_L$, namely the truncation of the square of the original infinite dimensional matrix K_1. We see here how the breakdown of the twist symmetry happens in this regularization. In the original representation before changing rows and columns into (3.12), the difference between truncating before squaring, $(K_1|_L)^2$, and truncating after squaring, $(K_1)^2|_L$, appears only at the last (L,L) component. As seen from (3.3), the last component of $(K_1|_L)^2$ is $(L-1)L$, while that of $(K_1)^2|_L$ is $(L-1)L+L(L+1)=2L^2$. Therefore, when $L=2\ell$, since the last component in the original representation belongs to the even-even block, the odd-odd block of the matrix $(K_1)^2|_{2\ell}$ should be the same as that of $(K_1|_{2\ell})^2$. As for the even-even block of $(K_1)^2|_{2\ell}$, it can be read off from another matrix $(K_1|_{2\ell+1})^2$. Note that, while the odd-odd block of this matrix $(K_1|_{2\ell+1})^2$ is enlarged to $(\ell+1)\times(\ell+1)$, its even-even block is of the same size $\ell\times\ell$ as the even-even block of $(K_1)^2|_{2\ell}$. Since the last $(2\ell+1,2\ell+1)$ component of $(K_1|_{2\ell+1})^2$ belongs to the odd-odd block, the even-even block of this matrix is the same as that of $(K_1)^2|_{2\ell}$, which is of our interest. Therefore the final expression of $(K_1)^2|_{2\ell}$ is given by

$$(K_1)^2|_{2\ell} = \begin{pmatrix} \left[(K_1|_{2\ell})^2\right]_{oo} & 0 \\ 0 & \left[(K_1|_{2\ell+1})^2\right]_{ee} \end{pmatrix} = \begin{pmatrix} P_{2\ell}\Lambda^2_{2\ell}P^T_{2\ell} & 0 \\ 0 & Q_{2\ell+1}\Lambda^2_{2\ell+1}Q^T_{2\ell+1} \end{pmatrix}. \tag{3.22}$$

Note that the degeneracy of eigenvalues between the odd-odd block and the even-even one is in fact lifted in the regularized expression (3.22). In the case $L=2\ell+1$, a similar argument gives

$$(K_1)^2|_{2\ell+1} = \begin{pmatrix} P_{2\ell+2}\Lambda^2_{2\ell+2}P^T_{2\ell+2} & 0 \\ 0 & Q_{2\ell+1}\Lambda^2_{2\ell+1}Q^T_{2\ell+1} \end{pmatrix}. \tag{3.23}$$

3.3 Evaluation of G

Having seen in the previous two subsections how we should regularize the singularity in the physical observables and how the degeneracy of the eigenvalues due to twist symmetry is lifted in this regularization, let us proceed to evaluating G analytically. For simplicity, in the following we shall take the truncation level L to be an even number. Noting that

$$(K_1\mathbf{u})_n = \begin{cases} 1\,, & n{=}1\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{3.24}$$

$$(K_1K_1\mathbf{u})_n = \begin{cases} -\sqrt{2}\,, & n{=}2\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{3.25}$$

which can be easily seen from (3.3) and (3.7), we can further simplify the expression (3.11) into

$$\begin{aligned} G &= \frac{\pi}{4}\left(\frac{1}{\sqrt{K_1^2}} - K_1\left(\frac{1}{\sqrt{K_1^2}}\right)^3 K_1\right)[1,1] \\ &= \frac{\pi}{4}\left(\frac{1}{\sqrt{K_1^2}}[1,1] - 2\left(\frac{1}{\sqrt{K_1^2}}\right)^3[2,2]\right). \end{aligned} \tag{3.26}$$

Here $M[n,m]$ for a generic matrix stands for its (n,m) component $M_{n,m}$. The component indices in (3.24)–(3.26) are those in the original representation before changing to the representation of (3.12).

It is by no means an easy task to find one of the components of the matrices in (3.26) analytically. However, since the eigenvalue problem of the matrix K_1 is solved in [11], we can evaluate G by using it. The eigenvector $\mathbf{f}^{(\kappa)}$ of K_1 corresponding to the eigenvalue κ,

$$K_1\mathbf{f}^{(\kappa)} = \kappa\mathbf{f}^{(\kappa)}\,, \tag{3.27}$$

is given by (B.3). We define the twist-odd eigenvectors $\mathbf{p}_n$ and twist-even eigenvectors $\mathbf{q}_n$ of K_1^2 as

$$(\mathbf{p}_n)_m = \frac{1}{2}\left(\mathbf{f}^{(\kappa_n)} + \mathbf{f}^{(-\kappa_n)}\right)_{2m-1} = \left(1, \frac{\kappa_n^2-2}{2\sqrt{3}}, \cdots\right), \tag{3.28}$$

$$(\mathbf{q}_n)_m = \frac{1}{2}\left(\mathbf{f}^{(\kappa_n)} - \mathbf{f}^{(-\kappa_n)}\right)_{2m} = \left(-\frac{\kappa_n}{\sqrt{2}}, -\frac{\kappa_n^3-8\kappa_n}{12}, \cdots\right), \tag{3.29}$$

for $\kappa_n = 2\pi n/\ln L$ with integer n and similar ones $\mathbf{p}_{n-\frac{1}{2}}$ and $\mathbf{q}_{n-\frac{1}{2}}$ for $\kappa_{n-\frac{1}{2}} = 2\pi(n-\frac{1}{2})/\ln L$. The matrices $P_{2\ell}$, $P_{2\ell+1}$, $Q_{2\ell}$ and $Q_{2\ell+1}$ are expressed using these eigenvectors. Introducing new symbols for these matrices to avoid cumbersome subscripts, we have

$$P_{\mathrm{H}} \equiv P_{2\ell} = \left(\overline{\mathbf{p}}_{1/2}, \overline{\mathbf{p}}_{3/2}, \cdots\right), \tag{3.30}$$

$$P_{\mathrm{I}} \equiv P_{2\ell+1}|_{\text{zero-mode removed}} = \left(\overline{\mathbf{p}}_1, \overline{\mathbf{p}}_2, \ldots\right), \tag{3.31}$$

$$Q_{\mathrm{H}} \equiv Q_{2\ell} = \left(\overline{\mathbf{q}}_{1/2}, \overline{\mathbf{q}}_{3/2}, \cdots\right), \tag{3.32}$$

$$Q_{\mathrm{I}} \equiv Q_{2\ell+1} = \left(\overline{\mathbf{q}}_1, \overline{\mathbf{q}}_2, \ldots\right), \tag{3.33}$$

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where the vectors with a bar, $\overline{\mathbf{p}}$ and $\overline{\mathbf{q}}$, denote the normalized ones of $\mathbf{p}$ and $\mathbf{q}$. The subscripts H and I imply half-an-odd integer and integer eigenvalues (in unit of $2\pi/\ln L$), respectively. We have defined P_{I} as $P_{2\ell+1}$ with the eigenvector corresponding to the zero eigenvalue removed.

Using the expression of $1/\sqrt{K_1^2}$,

$$\frac{1}{\sqrt{K_1^2}} = \begin{pmatrix} P_{\mathrm{H}}\Lambda_{\mathrm{H}}^{-1}P_{\mathrm{H}}^T & 0 \\ 0 & Q_{\mathrm{I}}\Lambda_{\mathrm{I}}^{-1}Q_{\mathrm{I}}^T \end{pmatrix}, \tag{3.34}$$

where Λ_{H} and Λ_{I} are the diagonal matrices of eigenvalues

$$\begin{aligned} \Lambda_{\mathrm{H}} &\equiv \Lambda_{2\ell} = \mathrm{diag}\left(\kappa_{n-\frac{1}{2}}\right), \\ \Lambda_{\mathrm{I}} &\equiv \Lambda_{2\ell+1} = \mathrm{diag}\left(\kappa_n\right), \end{aligned} \tag{3.35}$$

eq. (3.26) is rewritten into

$$\begin{aligned} G &= \frac{\pi}{4}\left(\sum_n \left(\left(\overline{\mathbf{p}}_{n-\frac{1}{2}}\right)_1\right)^2 \frac{1}{\kappa_{n-\frac{1}{2}}} - \sum_n \left((\overline{\mathbf{q}}_n)_1\right)^2 \frac{2}{\kappa_n^3}\right) \\ &= \frac{\pi}{4}\left(\sum_n \frac{1}{|\mathbf{p}_{n-\frac{1}{2}}|^2}\frac{1}{\kappa_{n-\frac{1}{2}}} - \sum_n \frac{(-\kappa_n/\sqrt{2})^2}{|\mathbf{q}_n|^2}\frac{2}{\kappa_n^3}\right), \end{aligned} \tag{3.36}$$

where $|\mathbf{a}|$ denotes the norm of a vector $\mathbf{a}$.

The norm of the eigenvectors is derived in appendix B. Especially for the vector in the finite L regularization, the norm is given by (B.15) with the delta function (B.14). Hence, from the definition of our eigenvectors $\mathbf{p}_{n-\frac{1}{2}}$ and $\mathbf{q}_n$, (3.28) and (3.29), we find that their norms are

$$\begin{aligned} |\mathbf{p}_{n-\frac{1}{2}}|^2 &= \frac{\delta(0)\sinh\left(\kappa_{n-\frac{1}{2}}\pi/2\right)}{\kappa_{n-\frac{1}{2}}}, \\ |\mathbf{q}_n|^2 &= \frac{\delta(0)\sinh\left(\kappa_n\pi/2\right)}{\kappa_n}, \end{aligned} \tag{3.37}$$

with $\delta(0) = \ln L/(2\pi)$ in the finite L regularization. Therefore our expression for G is reduced to

$$G = \frac{\pi}{4\delta(0)}\left(\sum_{n=1}^{L/2}\frac{1}{\sinh\left(\kappa_{n-\frac{1}{2}}\pi/2\right)} - \sum_{n=1}^{L/2}\frac{1}{\sinh\left(\kappa_n\pi/2\right)}\right). \tag{3.38}$$

In the limit $L\to\infty$, we can replace $\sinh x$ by x in (3.38) and finally obtain the desired result:[3]

$$G = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n-1/2} - \frac{1}{n}\right) = \ln 2\,. \tag{3.39}$$

The reason why we have obtained a non-vanishing value of G is that the degeneracy of eigenvalues between odd-odd and even-even sectors is lifted in the finite L regularization as seen from (3.22) and (3.34). As remarked below (3.17), the degeneracy is a general property of twist-odd symmetric matrices. Therefore, the phenomenon that a quantity such as G vanishing naively owing to the degeneracy acquires a non-zero value was called twist anomaly in [8]. Note also that the non-vanishing result of (3.39) comes only from infinitesimally small eigenvalues of order $1/\ln L$ in the limit $L \to \infty$. This should be regarded as a precise expression of (3.17) of [8] which has contribution only from the zero eigenvalue $\kappa = 0$ ($M_0 = -1/3$).

3.4 Properties of P and Q

In this subsection we shall derive a number of properties of the matrices $P_{\rm H}$, $P_{\rm I}$, $Q_{\rm H}$ and $Q_{\rm I}$ defined by (3.30)–(3.33). These properties are useful for systematic evaluation of the observables. In the last part of this subsection we shall rederive (3.39) for G by using the properties.

First, as can be seen from the inner product formula (B.13) (with $\sinh(\lambda\pi/2)$ approximated by $\lambda\pi/2$), the vectors $\mathbf{p}_n$ (3.28), $\mathbf{q}_n$ (3.29) and their half-an-odd counterparts satisfy the orthogonality,

$$\mathbf{p}_n\cdot\mathbf{p}_m = \mathbf{q}_n\cdot\mathbf{q}_m = \mathbf{p}_{n-\frac{1}{2}}\cdot\mathbf{p}_{m-\frac{1}{2}} = \mathbf{q}_{n-\frac{1}{2}}\cdot\mathbf{q}_{m-\frac{1}{2}} = \frac{\ln L}{4}\,\delta_{n,m}\,. \tag{3.40}$$

Corresponding to this fact, the three matrices $P_{\rm H}$, $Q_{\rm H}$ and $Q_{\rm I}$ are orthogonal ones:

$$\mathcal{O}^T\mathcal{O} = \mathcal{O}\mathcal{O}^T = 1\,, \qquad (\mathcal{O} = P_{\rm H}, Q_{\rm H}, Q_{\rm I})\,. \tag{3.41}$$

However, since the zero-mode is removed from the matrix $P_{\rm I}$ (3.31), though we have

$$P_{\rm I}^T P_{\rm I} = 1\,, \tag{3.42}$$

the completeness relation $P_{\rm I}P_{\rm I}^T = 1$ does not hold.

To derive the formulas associated with $P_{\rm I}$, let us consider the products $P_{\rm H}^T P_{\rm I}$ and $Q_{\rm H}^T Q_{\rm I}$. Their components are calculated by using (B.13) to be given by[4]

$$\left(P_{\rm H}^T P_{\rm I}\right)_{n,m} = \frac{4}{\ln L}\,\mathbf{p}_{n-\frac{1}{2}}\cdot\mathbf{p}_m = -\left(n-\frac{1}{2}\right)D_{n,m}\,, \tag{3.43}$$

[3]The replacement of $\sinh x$ with x is justified as follows. We divide the interval of the summation in (3.38) into two parts, $1 \le n \le \ln L/\pi^2$ and $\ln L/\pi^2 < n \le L/2$. First, the two summations in (3.38) contributed from the interval $\ln L/\pi^2 < n \le L/2$ cancel each other in the limit $L \to \infty$, because both of them converge to the same and well-defined integral $\int_1^\infty dx/2\sinh x$. Second, we can show that the summation in (3.38) from the interval $1 \le n \le \ln L/\pi^2$ is bounded from below by the same summation with $\sinh x$ replaced by x, using the fact that $1/x - 1/\sinh x$ is a monotonously increasing function in $0 < x < 1$. The upper bound is obtained similarly by using the expression of (3.38) with κ_n paired with $\kappa_{n+1/2}$ instead of the original $\kappa_{n-1/2}$. Both the upper and the lower bounds for (3.38) are reduced to (3.39) as $L \to \infty$.

[4]Strictly speaking, the inner product $\mathbf{p}_{n-\frac{1}{2}}\cdot\mathbf{p}_m$ between the ℓ component vector $\mathbf{p}_{n-\frac{1}{2}}$ and the $(\ell+1)$ one $\mathbf{p}_m$ is defined by removing the $(\ell+1)$-th component of $\mathbf{p}_m$. Equivalently, the matrix product $P_{\rm H}^T P_{\rm I}$ should be understood to imply $P_{\rm H}^T(1_{\ell\times\ell}, \mathbf{0})P_{\rm I}$.

$$\left(Q_{\mathrm{H}}^T Q_{\mathrm{I}}\right)_{n,m} = \frac{4}{\ln L}\,\mathbf{q}_{n-\frac{1}{2}} \cdot \mathbf{q}_m = -D_{n,m}\, m\,, \tag{3.44}$$

where $D_{n,m}$ is defined by

$$D_{n,m} = \frac{2}{\pi}\frac{(-1)^{n+m}}{(n-1/2)^2 - m^2}\,. \tag{3.45}$$

Then, the following formula is an immediate consequence of (3.43) and (3.44):

$$P_{\mathrm{H}}^T P_{\mathrm{I}} \Lambda_{\mathrm{I}} = \Lambda_{\mathrm{H}} Q_{\mathrm{H}}^T Q_{\mathrm{I}}\,. \tag{3.46}$$

As seen from (3.18) rewritten in the present notation as

$$K_1|_{2\ell} = \begin{pmatrix} 0 & P_{\mathrm{H}}\Lambda_{\mathrm{H}} Q_{\mathrm{H}}^T \\ Q_{\mathrm{H}}\Lambda_{\mathrm{H}} P_{\mathrm{H}}^T & 0 \end{pmatrix}, \tag{3.47}$$

eq. (3.46) multiplied by P_{H} on the left, $P_{\mathrm{H}}\Lambda_{\mathrm{H}} Q_{\mathrm{H}}^T Q_{\mathrm{I}} = P_{\mathrm{I}}\Lambda_{\mathrm{I}}$, just corresponds to the relation $K_1 \mathbf{q}_n = \kappa_n \mathbf{p}_n$ following from (3.27).[5]

Let us mention another important formula concerning P_{I}:

$$P_{\mathrm{H}}^T P_{\mathrm{I}} \left(P_{\mathrm{H}}^T P_{\mathrm{I}}\right)^T = 1 - W\,, \tag{3.48}$$

where the matrix W on the r.h.s. is

$$W_{nm} = \frac{2}{\pi^2}\frac{(-1)^{n+m}}{(n-1/2)(m-1/2)}\,. \tag{3.49}$$

Equation (3.48) is easily proved from (3.43). As seen from the direct product form of W (3.49), it is a projection operator of rank one:

$$W^2 = W\,, \qquad \operatorname{tr} W = 1\,. \tag{3.50}$$

Using $P_{\mathrm{H}} P_{\mathrm{H}}^T = 1$, eq. (3.48) is rewritten into

$$P_{\mathrm{I}} P_{\mathrm{I}}^T = 1 - P_{\mathrm{H}} W P_{\mathrm{H}}^T\,. \tag{3.51}$$

Having finished the derivation of the formulas of P and Q, let us turn to a recalculation of G. Using the matrix representation (3.34) and (3.47) and the fact that the $[1,1]$ component has contribution only from the odd-odd sector, we can rewrite the first expression of (3.26) for G in terms of P and Q as

$$G = \frac{\pi}{4}\left(P_{\mathrm{H}}\Lambda_{\mathrm{H}}^{-1} P_{\mathrm{H}}^T - P_{\mathrm{H}}\Lambda_{\mathrm{H}} Q_{\mathrm{H}}^T Q_{\mathrm{I}} \Lambda_{\mathrm{I}}^{-3} Q_{\mathrm{I}}^T Q_{\mathrm{H}}\Lambda_{\mathrm{H}} P_{\mathrm{H}}^T\right)[1,1]\,. \tag{3.52}$$

Using (3.46) and its transpose, eq. (3.52) can be brought to an expression without Q:

$$\begin{aligned} G &= \frac{\pi}{4}\left(P_{\mathrm{H}}\Lambda_{\mathrm{H}}^{-1} P_{\mathrm{H}}^T - P_{\mathrm{I}}\Lambda_{\mathrm{I}}^{-1} P_{\mathrm{I}}^T\right)[1,1] \\ &= \frac{\pi}{\ln L}\sum_{n=1}^{\infty}\left(\frac{1}{\kappa_{n-\frac{1}{2}}} - \frac{1}{\kappa_n}\right), \end{aligned} \tag{3.53}$$

where we have used that $(\overline{\mathbf{p}}_n)_1 = (\overline{\mathbf{p}}_{n-\frac{1}{2}})_1 = 2/\sqrt{\ln L}$. Equation (3.53) is nothing but the previous (3.39).

[5] Among the other three relations following from (3.27), the two corresponding to $K_1 \mathbf{p}_{n-\frac{1}{2}} = \kappa_{n-\frac{1}{2}} \mathbf{q}_{n-\frac{1}{2}}$ and the one with $\mathbf{p}$ and $\mathbf{q}$ exchanged are trivial consequences of (3.41). However, the remaining one, $Q_{\mathrm{H}}\Lambda_{\mathrm{H}} P_{\mathrm{H}}^T P_{\mathrm{I}} = Q_{\mathrm{I}}\Lambda_{\mathrm{I}}$ corresponding to $K_1 \mathbf{p}_n = \kappa_n \mathbf{q}_n$, does not hold in the present regularization.

4. Energy density

Now let us proceed to the evaluation of H. Expressing M_0, M_1 and $\mathbf{v}_1$ in H (2.34) in terms of K_1 and $\mathbf{u}$ by using (3.8)–(3.10) and keeping only those terms with degree of divergence equal to three, we get

$$H = \frac{\pi}{6}\,\mathbf{u}^T K_1 \left\{ \frac{1}{\sqrt{K_1^2}}\mathcal{R} - 2\,\frac{1}{\sqrt{K_1^2}}K_1\frac{1}{\sqrt{K_1^2}}\mathcal{R}\frac{1}{\sqrt{K_1^2}}K_1 - \right.$$
$$\left. - 3\,K_1\left(\frac{1}{\sqrt{K_1^2}}\right)^2\mathcal{R}\frac{1}{\sqrt{K_1^2}}K_1 + 3\,K_1\left(\frac{1}{\sqrt{K_1^2}}\right)^3 K_1 \right\} K_1\mathbf{u}\,, \tag{4.1}$$

with $\mathcal{R}$ for (4.1) given by

$$\mathcal{R} = \left(1 + \frac{1}{3}K_1\frac{1}{\sqrt{K_1^2}}K_1\frac{1}{\sqrt{K_1^2}}\right)^{-1}. \tag{4.2}$$

Using that the second term and the sum of the last two terms in the curly bracket of (4.1) are rewritten, respectively, into

$$-\frac{1}{\sqrt{K_1^2}}\mathcal{R}K_1\left(\frac{1}{\sqrt{K_1^2}}\right)^2 K_1 - \left(K_1\left(\frac{1}{\sqrt{K_1^2}}\right)^2 K_1\frac{1}{\sqrt{K_1^2}}\mathcal{R}\right)^T, \tag{4.3}$$

and

$$K_1\left(\frac{1}{\sqrt{K_1^2}}\right)^2 K_1\frac{1}{\sqrt{K_1^2}}\mathcal{R}K_1\left(\frac{1}{\sqrt{K_1^2}}\right)^2 K_1\,, \tag{4.4}$$

we obtain a simpler expression of H:

$$H = \frac{\pi}{6}\left(1 - K_1\left(\frac{1}{\sqrt{K_1^2}}\right)^2 K_1\right)\frac{1}{\sqrt{K_1^2}}\mathcal{R}\left(1 - K_1\left(\frac{1}{\sqrt{K_1^2}}\right)^2 K_1\right)[1,1]\,. \tag{4.5}$$

Now we use the matrix representations (3.34) and (3.47) for (4.5). First, we have

$$\begin{aligned}\left(1 - K_1\left(\frac{1}{\sqrt{K_1^2}}\right)^2 K_1\right)_{oo} &= 1 - P_{\mathrm{H}}\Lambda_{\mathrm{H}}Q_{\mathrm{H}}^T Q_{\mathrm{I}}\Lambda_{\mathrm{I}}^{-2}Q_{\mathrm{I}}^T Q_{\mathrm{H}}\Lambda_{\mathrm{H}}P_{\mathrm{H}}^T \\ &= 1 - P_{\mathrm{I}}P_{\mathrm{I}}^T \\ &= P_{\mathrm{H}}WP_{\mathrm{H}}^T\,,\end{aligned} \tag{4.6}$$

where we have used (3.46) at the second equality, and (3.51) in obtaining the last expression. Next, using (3.46) we have

$$\begin{aligned}\left(\frac{1}{\sqrt{K_1^2}}\mathcal{R}\right)_{oo} &= P_{\mathrm{H}}\Lambda_{\mathrm{H}}^{-1}P_{\mathrm{H}}^T\left(1 + \frac{1}{3}\,P_{\mathrm{H}}\Lambda_{\mathrm{H}}Q_{\mathrm{H}}^T Q_{\mathrm{I}}\Lambda_{\mathrm{I}}^{-1}Q_{\mathrm{I}}^T Q_{\mathrm{H}}\Lambda_{\mathrm{H}}P_{\mathrm{H}}^T P_{\mathrm{H}}\Lambda_{\mathrm{H}}^{-1}P_{\mathrm{H}}^T\right)^{-1} \\ &= P_{\mathrm{H}}\Lambda_{\mathrm{H}}^{-1}P_{\mathrm{H}}^T\left(1 + \frac{1}{3}\,P_{\mathrm{I}}Q_{\mathrm{I}}^T Q_{\mathrm{H}}P_{\mathrm{H}}^T\right)^{-1}.\end{aligned} \tag{4.7}$$

L	$H/[\ln(27/16)/2]$
50	0.9999481903
100	0.9999869824
150	0.9999942047
200	0.9999967374
250	0.9999979109
300	0.9999985488

Table 2: Numerical values of H (4.10) for various cutoff L.

Therefore, H of (4.5) is rewritten into

$$H = \frac{\pi}{6} P_{\mathrm{H}} W \Lambda_{\mathrm{H}}^{-1} \left(1 + \frac{1}{3} P_{\mathrm{H}}^T P_{\mathrm{I}} Q_{\mathrm{I}}^T Q_{\mathrm{H}} \right)^{-1} W P_{\mathrm{H}}^T [1,1] . \tag{4.8}$$

Then, using (3.43), (3.44) and

$$\left(W P_{\mathrm{H}}^T \right)_{n1} = \sum_{m=1}^{\infty} W_{nm} \frac{2}{\sqrt{\ln L}} = -\frac{2}{\sqrt{\ln L}} \frac{(-1)^n}{\pi (n - 1/2)} , \tag{4.9}$$

we obtain the final expression of H:

$$H = \frac{1}{3\pi^2} \sum_{n,m=1}^{\infty} \frac{1}{(n-1/2)^2} \left(A^{-1} \right)_{nm} \frac{1}{m - 1/2} , \tag{4.10}$$

where the matrix A is given by

$$A_{nm} = \delta_{n,m} + \frac{4}{3\pi^2} \left(n - \frac{1}{2} \right) \sum_{k=1}^{\infty} \frac{k}{[(n-1/2)^2 - k^2]\, [(m-1/2)^2 - k^2]} . \tag{4.11}$$

Now we have obtained a largely simplified expression of H compared with the original (2.34) or (4.1). The main difference between the original expression (4.1) for H and the present one (4.10) is that, although each term in (4.1) does contain divergence and they cancel as a whole, our final expression (4.10) is a well-defined infinite series without containing any divergences. Unfortunately, we have not succeeded in evaluating the infinite series analytically. Instead, we have carried out numerical calculation of (4.10) by reintroducing the cutoff L to the infinite summations in (4.10) and (4.11). The result given in table 2 suggests very strongly that $H = (1/2) \ln(27/16)$, in agreement with the result of [10]. Therefore, the interpretation of the present classical solution Ψ_{c} as the configuration of two D25-branes [8] is rejected.

5. Conclusion

In this paper we have shown how the twist anomaly is evaluated analytically. During our analysis, we found how the twist anomaly is realized in the finite-size matrix regularization. Naively, the eigenvalues of the Neumann coefficient matrix M_0 degenerate due to the twist symmetry. However, after introducing the regularization, the twist symmetry breaks down

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and the degeneracy of the eigenvalues is lifted. The quantity G was evaluated exactly to reproduce the expected tachyon mass squared of -1. On the other hand, for the quantity H related to the energy density, we obtained a simple expression as a well-defined infinite series, though we could not evaluate its value analytically. We shall conclude our paper by presenting several further directions of our analysis.

- As a technical problem, we have to evaluate the infinite series (4.10) for H analytically. Proof of the eigenvalue spectrum (3.17) of $K_1|_{2\ell}$, which is an assumption in this paper, is also a remaining subject. It is also necessary to give a rigorous justification to the prescription of keeping only terms with degree of divergence equal to three (see section 2.2).

- In the usual terminology, anomaly appears when there are no regularizations compatible with all the symmetries. We would like to understand the twist anomaly in the same sense. Especially, it should be important to understand which symmetries our finite-size matrix regularization respects.

- We evaluated the twist anomaly by adopting the regularization of truncating the size of the infinite matrices. We would also like to derive the same results as obtained in this paper by a more refined and more systematic regularization. Methods of [13] would be an interesting possibility.

- In [10] and our present analysis, it was found that the energy density of the solution Ψ_{c} does not reproduce the correct D25-brane tension. In [10] it was further pointed out that the reason why the energy density deviates from the expected value is that the present tachyon wave function Φ_{t} does not satisfy the linearized equation of motion $\mathcal{Q}_{\mathrm{B}}\Phi_{\mathrm{t}} = 0$ (2.22) in the strong sense. Having seen the correspondence of the final results between our algebraic analysis [6, 8] and the geometrical approach [10], it is an urgent task to obtain the tachyon wave function which satisfy the linearized equation of motion in the strong sense and at the same time reproduces the correct D25-brane tension. A root of the problem in the geometric approach lies in the fact that the cubic product $\Phi_1 \cdot (\Phi_2 * \Phi_3)$ of three sliver-based states Φ_k with momentum insertion depends on how we take the limit $n_k \to \infty$ [10, 21], where we express the state Φ_k as n_k-wedge state [20].

- As far as we have investigated, physical observables in VSFT are always related to quantities that naively vanish. We would like to understand the deep reason of this phenomenon. It might be related to the fact that in VSFT (2.1) expanded around $\Phi = 0$ there are no physical excitations since the kinetic term consists only of the purely ghost operator.

- Note that the twist anomaly are written in terms of the Neumann coefficients, which express the open string interactions. This indicates that the twist anomaly might be a fundamental phenomenon which appears universally in the open string interactions. Usually the scattering amplitudes in CSFT have been calculated by mapping them to a complex plane. However, recalling that the twist anomaly can also be evaluated on the complex plane [10], the string amplitudes calculated thoroughly in terms of the Neumann coefficients might be recognized as twist anomaly (though this expectation contradicts the one mentioned in the above item). We would also like to know how the recent explanation of the emergence of closed strings in VSFT [18] is related to twist anomaly.

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A. Proof of the vector formulas (3.5) and (3.6)

In this appendix, we present a proof of the formulas (3.5) and (3.6) which express $\mathbf{v}_0$ and $\mathbf{v}_1$ in terms of a simpler vector $\mathbf{u}$. Since (3.5) is not directly used in this paper and its proof is almost the same as that for (3.6), we shall mainly concentrate on proof of the formula (3.6). Our argument is similar to that for the eigenvector of M_0 corresponding to the eigenvalue $-1/3$ given in [11, section 3].

The original matter Neumann coefficients have the following integral representation [22, 23]:

$$V_{nm}^{rs} = -\frac{1}{\sqrt{nm}} \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^n w^m} \frac{f_r'(z) f_s'(w)}{(f_r(z) - f_s(w))^2}\,, \tag{A.1}$$

$$V_{n0}^{rs} = -\frac{1}{\sqrt{n}} \oint \frac{dz}{2\pi i} \frac{1}{z^n} \frac{f_r'(z)}{f_r(z) - f_s(0)}\,, \tag{A.2}$$

where $f_r(z)$ is given by[6]

$$f_k(z) = f(z)\, \omega^{-k}\,, \tag{A.3}$$

with

$$f(z) = \left(\frac{1+iz}{1-iz}\right)^{2/3}, \qquad \omega = e^{2\pi i/3}\,. \tag{A.4}$$

The integration contours in (A.1) and (A.2) are circles around the origin. These Neumann coefficient matrices are related to our present ones by[7]

$$(CM_0)_{nm} = V_{nm}^{rr}\,, \qquad (CM_1)_{nm} = (V^{r,r+1} - V^{r,r-1})_{nm}\,, \tag{A.5}$$

$$(\mathbf{v}_0)_n = \frac{1}{3}\left(2V^{rr} - V^{r,r+1} - V^{r,r-1}\right)_{n0}\,, \quad (\mathbf{v}_1)_n = (V^{r,r+1} - V^{r,r-1})_{n0}\,. \tag{A.6}$$

[6] Note that, compared with the formulas in [22, 23], ω is replaced with $1/\omega$.

[7] The Neumann coefficient V_{n0}^{rs} is unique only when the index s is contracted with a conserved quantity α_s satisfying $\sum_{s=1}^{3} \alpha_s = 0$. The vector $\mathbf{v}_0$ in this paper in a generic representation of V_{n0}^{rs} is $-(1/3)(\mathbf{v}_{+0} + \mathbf{v}_{-0})$ in [6], namely, the one defined by (A.6). This is a representation independent quantity. Only when V_{n0}^{rs} is defined through the 6-string Neumann coefficient [15, 16], we have $(\mathbf{v}_0)_n = V_{n0}^{rr}$ since (2.6), i.e. $(V^{rr} + V^{r,r+1} + V^{r,r-1})_{n0} = 0$, holds in this representation. In the text we are taking the representation in terms of the 6-string Neumann coefficient.

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Especially, the integral representation of CM_1 and $\mathbf{v}_1$ are given as

$$
\begin{aligned}
(CM_1)_{nm} &= -\frac{4i}{3}\sqrt{\frac{m}{n}}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{f(z)}{z^n w^{m+1}(1+z^2)}\times \\
&\qquad\qquad\times\left[\frac{1}{f(z)-\omega^* f(w)}-\frac{1}{f(z)-\omega f(w)}\right], &\text{(A.7)}\\
(\mathbf{v}_1)_n &= -\frac{4i}{3}\frac{1}{\sqrt{n}}\oint\frac{dz}{2\pi i}\frac{f(z)}{z^n(1+z^2)}\left[\frac{1}{f(z)-\omega^*}-\frac{1}{f(z)-\omega}\right]. &\text{(A.8)}
\end{aligned}
$$

In deriving (A.7) we have carried out an integration by parts with respect to w. To prove the formula (3.6), let us calculate $M_1\mathbf{u}$. Since we have $M_1\mathbf{u} = -CM_1\mathbf{u}$ owing to the twist property $CM_1C = -M_1$ and $C\mathbf{u} = \mathbf{u}$, we shall calculate $-CM_1\mathbf{u}$.

To make the following calculation well-defined, we use the regularized version of $\mathbf{u}$ instead of the original one (3.7):

$$
q_n = \frac{a^{-n-1}}{2\sqrt{n}}\left[i^n + (-i)^n\right], \qquad (a\to 1+0)\,. \tag{A.9}
$$

Then, from (A.7), (A.9) and the geometric series

$$
\sum_{m=1}^{\infty}\frac{i^m+(-i)^m}{2\,(aw)^{m+1}} = -\frac{1}{aw(1+a^2w^2)}\,, \tag{A.10}
$$

we obtain

$$
(M_1\mathbf{u})_n = -\frac{4i}{3}\frac{1}{\sqrt{n}}\oint\frac{dz}{2\pi i}\frac{f(z)}{z^n(1+z^2)}F_1(z)\,, \tag{A.11}
$$

with $F_1(z)$ defined by

$$
F_1(z) = \oint\frac{dw}{2\pi i}\frac{1}{aw(1+a^2w^2)}\left[\frac{1}{f(z)-\omega^* f(w)}-\frac{1}{f(z)-\omega f(w)}\right]. \tag{A.12}
$$

The integration contour for (A.12) must satisfy $|w| > 1/a$ due to the convergence requirement of the series (A.10). The integration $F_1(z)$ has contributions from poles at $w = 0$ and $\pm i/a$ (the pole at $w = -1/z$ corresponding to $f(z) - \omega^{\pm} f(w) = 0$ is outside the w-integration contour), and we have

$$
\begin{aligned}
aF_1(z) &= \frac{1}{f(z)-\omega^*}-\frac{1}{f(z)-\omega}+ \\
&\quad+\sum_{\pm}\frac{1}{(\pm i/a)2a^2(\pm i/a)}\left[\frac{1}{f(z)-\omega^* f(\pm i/a)}-\frac{1}{f(z)-\omega f(\pm i/a)}\right] \\
&\underset{a\to 1}{\longrightarrow}\ \frac{1}{f(z)-\omega^*}-\frac{1}{f(z)-\omega}\,. &\text{(A.13)}
\end{aligned}
$$

Note that both of the terms coming from $w = \pm i/a$ vanish in the limit $a \to 1$.

Comparing (A.11) for $M_1\mathbf{u}$ with $F_1(z)$ given by (A.13) and the integral representation (A.8) of $\mathbf{v}_1$, we find that they are equal:

$$
\mathbf{v}_1 = M_1\mathbf{u}\,. \tag{A.14}
$$

We can prove the formula (3.5) in quite a similar way.

B. Inner product of the eigenvectors

In this appendix we calculate the inner product of the eigenvectors of the matrix K_1.[8] The eigenvalue problem of the matrix K_1 has been solved in [11]. There, the matrix K_1 is represented as a differential operator $-(1+z^2)(d/dz)$ acting on the function $f(z)$ made from a generic vector $\mathbf{f} = (f_n)$:

$$f(z) = \sum_{n=1}^{\infty} \frac{f_n}{\sqrt{n}} z^n \,, \tag{B.1}$$

and the eigenvalues and the eigenvectors are obtained by solving differential equations. The eigenvalues of the matrix K_1 range over the real axis uniformly. The function $f^{(\kappa)}(z)$ corresponding to the eigenvector $\mathbf{f}^{(\kappa)}$ with eigenvalue κ is given by

$$f^{(\kappa)}(z) = \frac{1}{\kappa}\left(1 - \exp(-\kappa \tan^{-1} z)\right) = z - \frac{\kappa}{\sqrt{2}}\frac{z^2}{\sqrt{2}} + \frac{\kappa^2-2}{2\sqrt{3}}\frac{z^3}{\sqrt{3}} + \cdots \,. \tag{B.2}$$

From (B.2) the eigenvectors $\mathbf{f}^{(\kappa)}$ before normalization can be read off as

$$\mathbf{f}^{(\kappa)} = \left(1, -\frac{\kappa}{\sqrt{2}}, \frac{\kappa^2-2}{2\sqrt{3}}, \ldots\right), \tag{B.3}$$

The inner product between two generic vectors $\mathbf{f}$ and $\mathbf{g}$ is defined by

$$\mathbf{f}\cdot\mathbf{g} \equiv \sum_{n=1}^{\infty} f_n g_n \,. \tag{B.4}$$

It is expressed by a contour integral using the corresponding functions $f(z)$ and $g(z)$ as

$$\mathbf{f}\cdot\mathbf{g} = \oint_{|z|=1} \frac{dz}{2\pi i}\left(\frac{d}{dz} f(z)\right) g\left(\frac{1}{z}\right). \tag{B.5}$$

In particular, the inner product between the eigenvectors is given by

$$\mathbf{f}^{(\kappa)}\cdot\mathbf{f}^{(\lambda)} = \frac{-1}{\lambda}\oint_{|z|=1} \frac{dz}{2\pi i}\frac{1}{1+z^2}\exp\left(-\kappa \tan^{-1} z\right)\exp\left(-\lambda \tan^{-1}\frac{1}{z}\right). \tag{B.6}$$

However, this integral is not well-defined since there exist poles at $z = \pm i$ and some branch-cuts. In order to treat $\tan^{-1} z = (1/2i)\ln[(1+iz)/(1-iz)]$ properly, let us take the branch-cut of $\ln z$ to be $\Im z = 0$, $\Re z < 0$. Then the branch-cuts of $\tan^{-1} z$ and $\tan^{-1} 1/z$ runs over the imaginary axis and these cuts meet at $z = \pm i$. Therefore, we adopt the same regularization as used in (A.9). Namely, we deform the function $f^{(\kappa)}(z)$ into

$$f^{(\kappa)}(z) = \frac{1}{\kappa}\left(1 - \exp\left(-\kappa \tan^{-1}\frac{z}{a}\right)\right) = \sum_{n=1}^{\infty}\frac{f_n}{\sqrt{n}}\left(\frac{z}{a}\right)^n, \tag{B.7}$$

with $a = 1+0$. This deformation corresponds to replacing f_n with f_n/a^n, which serves effectively as truncation of the infinite dimensional vector into a finite L-dimensional one with

$$a^L \simeq e \,. \tag{B.8}$$

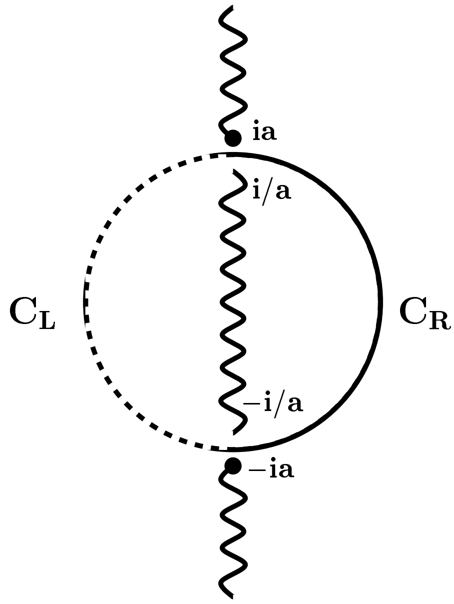

Figure 1: The contour of the integration (B.9) to evaluate the norm of $\mathbf{f}$. The blobs at $z = \pm ia$ are the poles of the integrand of (B.9).

On the other hand, this deformation slightly moves the poles and the endpoints of the branch-cuts as in figure 1. Since the branch-cuts of $\tan^{-1} z$ ($\tan^{-1} 1/z$) runs from $z = ia$ (i/a) to $z = -ia$ ($-i/a$) along the imaginary axis, the integral along the contour $|z| = 1$ is quite safe:

$$\mathbf{f}^{(\kappa)} \cdot \mathbf{f}^{(\lambda)} = \frac{-1}{\lambda} \oint_{|z|=1} \frac{dz}{2\pi i} \frac{a}{a^2+z^2} \exp\left(-\kappa \tan^{-1} \frac{z}{a}\right) \exp\left(-\lambda \tan^{-1} \frac{1}{az}\right) . \tag{B.9}$$

Next, we separate the contour of (B.9) into two segments C_R and C_L, with $C_{R(L)}$ being the parts of the original contour $|z| = 1$ on the right (left) half plane. Using the identity

$$\tan^{-1} z + \tan^{-1} \frac{1}{z} = \begin{cases} \pi/2 , & \Re z > 0 , \\ -\pi/2 , & \Re z < 0 , \end{cases} \tag{B.10}$$

we have

$$\begin{aligned} \mathbf{f}^{(\kappa)} \cdot \mathbf{f}^{(\lambda)} = \frac{-1}{\lambda} & \left[\exp\left(-\frac{\lambda\pi}{2}\right) \int_{C_R} \frac{dz}{2\pi i} + \exp\left(\frac{\lambda\pi}{2}\right) \int_{C_L} \frac{dz}{2\pi i} \right] \frac{a}{a^2+z^2} \times \\ & \times \exp\left(-(\kappa-\lambda) \tan^{-1} \frac{z}{a}\right) \exp\left(-\lambda \left(\tan^{-1} \frac{z}{a} - \tan^{-1} az\right)\right) . \end{aligned} \tag{B.11}$$

The difference

$$\tan^{-1} \frac{z}{a} - \tan^{-1} az , \tag{B.12}$$

[8]A similar derivation of the inner products has been given in [12]. However, since we need the inner product of finite-L truncated eigenvectors, we shall rederive the inner product in a form applicable to the calculations in the text.

vanishes[9] in the limit of $a \to 1$. Therefore, dropping the final exponential factor in (B.11), we obtain

$$\mathbf{f}^{(\kappa)} \cdot \mathbf{f}^{(\lambda)} = \frac{2\sinh(\lambda\pi/2)}{\lambda\pi} \frac{1}{\kappa - \lambda} \sin\left(\frac{\ln L}{2}(\kappa - \lambda)\right), \tag{B.13}$$

where we have used the relation (B.8) between a and L. If we further use one of definitions of the delta function:

$$\pi\delta(\kappa) = \lim_{L\to\infty} \frac{1}{\kappa} \sin\frac{\ln L}{2}\kappa\,, \tag{B.14}$$

we find that in the limit $L \to \infty$

$$\mathbf{f}^{(\kappa)} \cdot \mathbf{f}^{(\lambda)} = \frac{2\sinh(\lambda\pi/2)}{\lambda} \delta(\kappa - \lambda)\,. \tag{B.15}$$

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